BINARY RELATIONS ON SETS

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Given two sets A and B, the Cartesian product $A\times B$ of these sets is the set of ordered pairs of elements of A and B where the first element of the pair is from A and the second is from B. For example

$$\begin{aligned} A &= \{1,2,3\} \\ B &= \{3,5\} \end{aligned} \tag{1}$$

then

$$A\times B=[(1,3),(1,5),(2,3),(2,5),(3,3),(3,5)] \tag{2}$$

If we consider the Cartesian product $A\times A$ of a set with itself, then we can define a *binary relation* $\mathcal{R}$ on A as a subset of $A\times A$. Thus with

$$A\times A=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\} \tag{3}$$

the Cartesian product contains $3^2=9$ elements. A relation $\mathcal{R}$ on A can contain any subset of these 9 elements. Since each element may or may not be included in $\mathcal{R}$, there are a total of $2^9=512$ distinct relations on A. Note that the empty set $\emptyset$ is a relation on any set.

Despite the normal English meaning of the word 'relation', there doesn't have to be any specific relationship between the members of the ordered pairs in $\mathcal{R}$. However, in practical applications of relations, it's more usual to define relations where there is some actual relation between the elements. For example, we might define a relation $\mathcal{R}$ on the set $\mathbb{R}$ of real numbers as

$$\mathcal{R}=\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\right\} \tag{4}$$

Here, $\mathbb{R}^2$ means the Cartesian product of the real number set with itself, so that $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$. The elements of the ordered pair (x,y) are related by the equation $x^2+y^2=1$, so that they lie on the unit circle.

PROPERTIES OF RELATIONS ON SETS

We now consider the classification of types of relations on a set A. The relation $\mathcal{R}$ is:

- Reflexive, if $(a,a)\in\mathcal{R}$ for every $a\in A$.

- Symmetric, if for all $a, b \in A$, $(a,b) \in \mathcal{R} \Rightarrow (b,a) \in \mathcal{R}$. That is, every occurrence of an ordered pair must have another ordered pair with the elements swapped.
- Antisymmetric, if for all $a, b \in A$, $(a,b) \in \mathcal{R} \wedge (b,a) \in \mathcal{R} \Rightarrow a = b$. (The symbol $\wedge$ stands for the logical 'and'.) That is, the only time both (a,b) and (b,a) appear in $\mathcal{R}$ is when $a = b$.
- Transitive, if for all $a, b, c \in A$, $(a,b) \in \mathcal{R} \wedge (b,c) \in \mathcal{R} \Rightarrow (a,c) \in \mathcal{R}$.
- Connex, if for all $a, b \in A$, either $(a,b) \in \mathcal{R}$ or $(b,a) \in \mathcal{R}$ (or both). In particular, a connex relation requires $(a,a) \in \mathcal{R}$ for all $a \in A$, so a connex relation is also reflexive.

Example 1. Consider the relation

$$\mathcal{R} = \left\{(x,y) \in \mathbb{Z}^2 : x + y < 5\right\} \tag{5}$$

The set $\mathbb{Z}$ is the set of integers. The relation is not reflexive since there are many pairs for which $x + x \geq 5$. It is symmetric, since if $x + y < 5$, then $y + x < 5$. This also implies that it is not antisymmetric. It is not transitive, since $(3,0) \in \mathcal{R}$ and $(0,4) \in \mathcal{R}$, but $(3,4) \notin \mathcal{R}$. Since $\mathcal{R}$ is not reflexive, it is not a connex relation.

Example 2. Consider the relation

$$\mathcal{T} = \left\{(x,y) \in \mathbb{N}^2 : x + y > 1\right\} \tag{6}$$

The set $\mathbb{N}$ is the set of positive integers. Thus $x + y > 1$ for all values of x and y. In particular, $x + x > 1$ for all $x \in \mathbb{N}$, so $\mathcal{T}$ is reflexive. It is also symmetric, since $x + y > 1$ and $y + x > 1$ are both true. It is not antisymmetric, since $x + y > 1$ and $y + x > 1$ are both true for $x \neq y$. It is transitive, since if $x + y > 1$ and $y + z > 1$, then $x + z > 1$ as well. Since $(x,y) \in \mathcal{T}$ for all x and y, it is a connex relation.

Example 3. Consider the relation

$$\mathcal{V} = \left\{(x,y) \in \mathbb{Z}^2 : x + y \text{ is even}\right\} \tag{7}$$

Since $x + x = 2x$ is always even, the set is reflexive. It is symmetric since $x + y = y + x$, so if one is even, so is the other. It is not antisymmetric, since (x,y) and (y,x) are in $\mathcal{V}$ for many values where $x \neq y$. To see if it's transitive, consider the possibilities. If $x + y$ is even, then either x and y are both even, or they are both odd. Thus if $(x,y) \in \mathcal{V}$ and $(y,z) \in \mathcal{V}$, then either all of x, y and z are even or they are all odd. Therefore both x and z must be either both even or both odd, so $(x,z) \in \mathcal{V}$, and the relation is transitive. It is not connex, since $(x,y)\notin\mathcal{V}$ and $(y,x)\notin\mathcal{V}$ if x is even and y is odd.

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